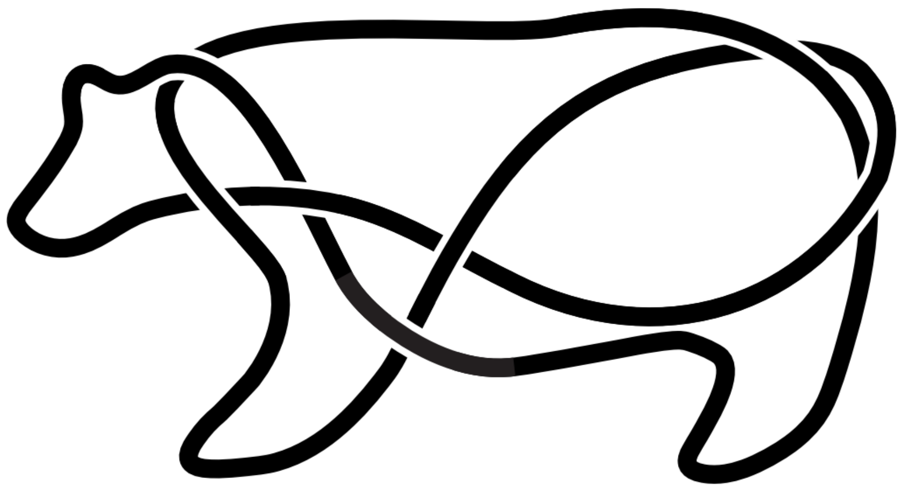
Berkeley Math Tournament 2025

Calculus Test



November 8, 2025

Time limit: 60 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators. Protractors, rulers, and compasses are permitted.

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2} + \frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.
- For fractions, both improper fractions and mixed numbers are acceptable.

1. Let $f(x) = x^2 - 11x + 31$. Distinct real numbers a and b satisfy $f(a) = f'(a)$ and $f(b) = f'(b)$. Given $a < b$, compute $10a + b$.

2. Determine the maximum value of
$$\frac{\left(2025^x\right)^\pi}{2025^{x^\pi}}$$
over all positive real numbers x.

3. Let $f(x) = \sqrt{3x^2 + 2x + 1}$. Compute
$$\lim_{x\to\infty} f'(x).$$

4. Compute
$$\lim_{n\to\infty} \left(\frac{2\tan^{-1}(n)}{\pi}\right)^n.$$

5. For each real number n, let $\lfloor n \rfloor$ denote the greatest integer less than or equal to n, and let $\lceil n \rceil$ denote the least integer greater than or equal to n. Compute
$$\int_0^\infty \frac{(-1)^{\lfloor x \rfloor}}{\lceil x \rceil^2 - \lfloor x \rfloor^2}\,\mathrm{d}x\,.$$

6. Evan wants to grow a garden, so he visits a florist selling n distinct flowers. He repeatedly rolls a fair n-sided die with faces labeled from 1 to n until he rolls a number other than 1. Evan then calculates N, the number of ways he can choose $\min\{k, n\}$ of the flowers, where k is the number of 1's rolled (possibly zero). Compute the limit as n goes to infinity of the expected value of N.

7. For positive real numbers a, let the circle $C(a)$ be the graph of $x^2+(y-a)^2 = 4a$ in the coordinate plane. Let $f(a)$ be the fraction of the area of $C(a)$ that lies above the parabola $y = x^2$. Find
$$\lim_{a\to\infty} f(a).$$

8. An integer k is chosen uniformly at random from the set $\{0, 1, \cdots, 2^{2025} - 1\}$. Compute the probability that
$$\int_0^\infty e^{-2025x} x^k \sin(2025x)\,\mathrm{d}x > 0.$$

9. Let $f(a, b) = (a^3 - 4a)(b - a) - \frac{(b-a)^2}{2}$. Find the least real number M such that for all finite, strictly increasing sequences $d_1, \ldots, d_n$ with $d_1 = -2$ and $d_n = 3$, the following inequality is satisfied:
$$\sum_{k=1}^{n-1} f(d_k, d_{k+1}) \le M.$$

10. Compute
$$\int_0^{\frac{\pi}{2}} \ln\big(\sin^4(x) + \cos^4(x)\big)\,\mathrm{d}x\,.$$