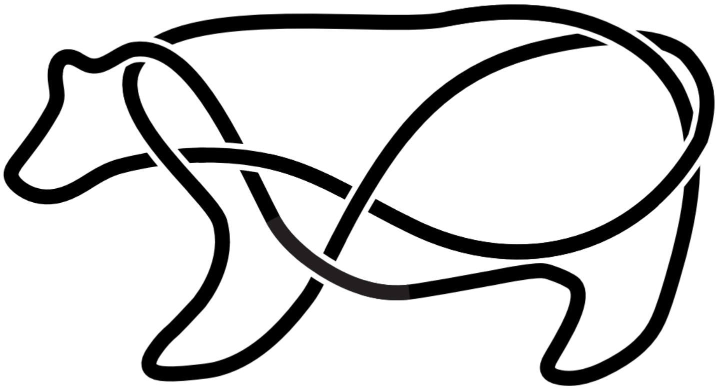
Berkeley Math Tournament 2025

Power Round



November 8, 2025

Time limit: 90 minutes.

Maximum score: 205 points.

Instructions: For this test, you will work in teams to solve multi-part, proof-oriented questions. Problems that use the words “compute,” “find,” or “draw” require only an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof.

Answers should be written on sheets of blank paper, clearly labeled, in order, with problem numbers in the top right corners. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your team envelope in order by problem number.

The problems are ordered by content, *not difficulty.* The difficulties of the problems are generally indicated by the point values assigned to them; it is to your advantage to attempt problems throughout the test. In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven’t solved them.

No calculators. Protractors, rulers, and compasses are permitted.

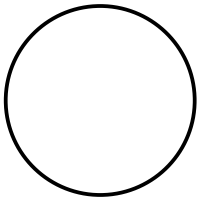
1 Introduction to Knots (31 pts)

Welcome to the BMT 2025 Power Round! In this round, we will explore the mathematics of knots. Understanding knots is useful for solving interesting problems in many fields, from protein folding to robotics. We will only have time to scratch the surface of this rich field, and we encourage you to explore it more on your own later.

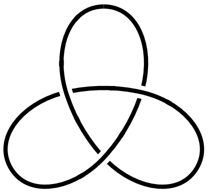
In our study of knots, the ends are connected to each other, so that each knot is a closed loop. You can think of a **knot** as the result of taking a physical string, tying it up as you want, and gluing the two loose ends together. A **knot diagram** (or **knot projection**), by contrast, is a two-dimensional drawing of a knot, recording what parts of the string cross over or under other parts. A *crossing* is where one part of the string passes over another part. In no part of the diagram may more than two lines cross at the same point. See below for examples of knot diagrams. Make sure you understand the distinction between knots and knot diagrams! They are not interchangeable.

The simplest possible knot is the *unknot*. Knots that can be transformed into the unknot are called *trivial knots* (see Question 1.5 for an example). These are loops without any crossings (a place where a strand of the knot crosses over another strand) that we cannot uncross, as shown below. We will think more about the idea of crossings that we cannot undo later. One possible knot diagram of the unknot is a circle with no crossings in the diagram, though there are others. Unless otherwise specified, the unknot refers to a knot or knot diagram with zero crossings. For the first part of this chapter, you may think of "transforming" a knot as manipulating the string in any way that doesn't break the closed loop formed by the string. Later, we will use a more rigorous definition of transformation called Reidemeister moves.

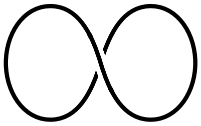
Another iconic knot is the trefoil knot, also shown below.



The unknot.



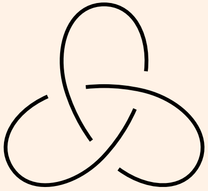
A trefoil.



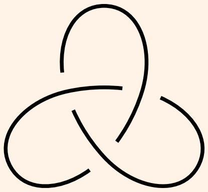
A trivial knot.

Question 1.1. (2 pts) Construct a trefoil knot and an unknot out of the provided pipe cleaners. Without breaking the loop, is it possible to transform the trefoil into the unknot? If it is possible, draw a sequence of diagrams demonstrating the transformation. If it is not possible, no explanation is required.

Question 1.2. (3 pts) Are the left-trefoil knot diagram and the right-trefoil knot diagram (shown below) two diagrams of the same knot? That is, is it possible to transform one into the other without breaking the loops? If it is possible, draw a sequence of diagrams demonstrating the transformation. If it is not possible, no explanation is required.



Left-trefoil knot



Right-trefoil knot

The above question illustrates an important topic in knot theory: equivalence.

How can we tell whether knots are equivalent? It turns out that there is not a simple answer. At this point, we haven't found general ways to determine that a given knot is nontrivial (it cannot be transformed into the unknot). To build up to this, we will first show that if a knot is nontrivial, then it must be at least as "complex" as a trefoil knot.

Question 1.3. (5 pts) Briefly explain how if a knot diagram has one crossing, the corresponding knot can be transformed into the unknot. (Hint: Draw the crossing first, and then make knot diagrams by connecting the ends in all possible ways so that no more crossings are created. Approach this task systematically so that your argument shows, without a doubt, that all possible knots have been considered.)

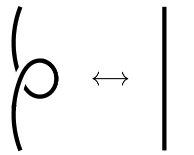
In order to show that a nontrivial knot is at least as complex as a trefoil knot, we need to show the same property holds for knots with two crossings: all of them can be transformed into the trivial knot. This can be shown with some more casework.

Thus, the simplest nontrivial knot must have at least 3 crossings. The trefoil knot has 3 crossings, but we haven't yet shown whether it is equivalent to the trivial knot. Proving that knots are nontrivial will take some extra machinery, which we'll develop more in Chapter 2.

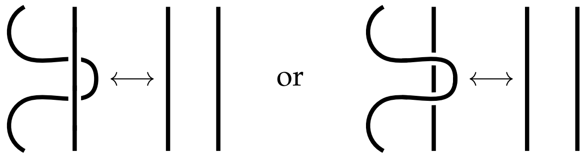
Now, let's return to an idea we've used in all of the previous problems: "transformations" of a knot. We want to make this idea more rigorous, so we can think about what happens to a knot under some transformation.

From now on, we'll define a transformation as a sequence of Reidemeister moves, which are ways to change a knot diagram that change its structure. When making a Reidemeister move, the diagram remains unchanged except for the change indicated by the move. There are three Reidemeister moves, as defined below.

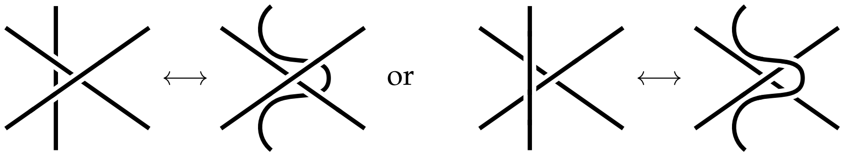
Definition 1.1. The **first Reidemeister move** is a simple twist or untwist of the knot.
The **second Reidemeister move** represents pulling a strand of the knot under or over another segment of string, creating or deleting two consecutive crossings.
The **third Reidemeister move** represents pulling a strand of the knot under or over another crossing, creating or deleting two consecutive crossings.



First Reidemeister move (R1)



Second Reidemeister move (R2)



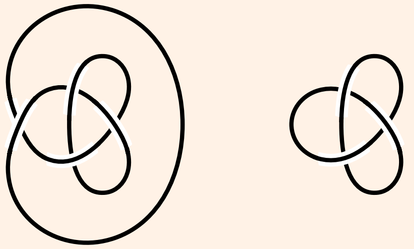
Third Reidemeister move (R3)

These moves might seem random, but it turns out that any valid transformation we make is actually a sequence of these Reidemeister moves! Even more importantly, the following theorem holds.

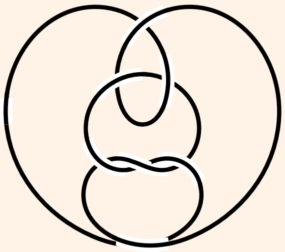
Theorem 1.2. (Reidemeister's Theorem): For any two diagrams of the same knot, there exists a sequence of Reidemeister moves that transforms one diagram into the other.

We won't prove this theorem, but its claim is incredible! We have three very simple moves we can perform on a knot diagram to get any other diagram of the same knot!

Question 1.4. (3 pts) Show that the two diagrams below represent the same knot by finding a sequence of Reidemeister moves from one to the other.

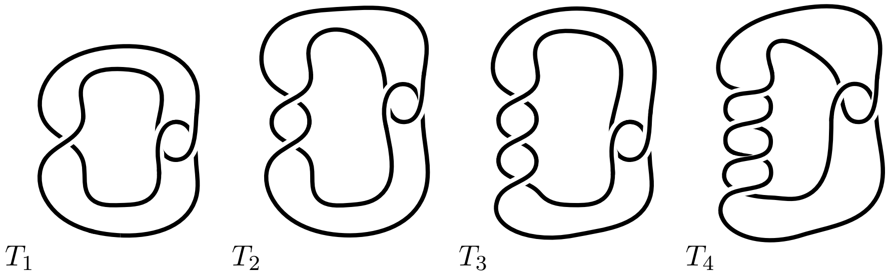


Question 1.5. (4 pts) The knot diagram shown below depicts the unknot. Find a sequence of Reidemeister moves that untangles it into the 0-crossing diagram shown before.



We will return to Reidemeister moves in the next section, but for now let's play a game! In the following game, called the Knotting-Unknotting Game, we begin with a knot diagram and hide the crossings. This will be the board for the game. On each turn, a player chooses an unresolved crossing and resolves it by setting it to either over or under (⤫ or ⤬). One player's goal is to create a nontrivial knot while the other player's goal is to create a trivial knot. Players alternate taking turns until all crossings are resolved. Player 1 goes first and Player 2 goes second.

Definition 1.3. A **twist knot** of n half-twists, denoted T_n, is obtained by twisting a loop n times and then connecting the ends together so that the knot is alternating, as shown below. T_n is nontrivial for all $n \geq 1$.



Question 1.6. (4 pts) For T_1, explain whether Player 1 or Player 2 has a winning strategy in the Knotting-Unknotting Game, assuming both players play optimally, if...

(a) Player 1's goal is to create the unknot.

(b) Player 1's goal is to create a nontrivial knot.

Question 1.7. (10 pts) Explain who has the winning strategy in the Knotting-Unknotting game played on T_n, which could depend on Player 1's goal and the value of n.

2 Invariants (35 pts)

So far, we've avoided the problem of determining whether two knots are equivalent– if one can be transformed into the other using Reidemeister moves. Let's tackle this problem using invariants.

Generally, an invariant is a property that doesn't change under some type of operation. For example, the number of fingers on my hand is invariant based on the order I count them in. We want to think about properties of knot diagrams that are invariant under Reidemeister moves since we noted before that all diagrams of a knot are equivalent up to these moves.

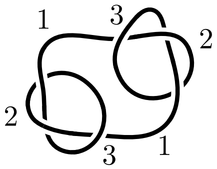
Definition 2.1. A **knot invariant** is a property computed on a knot with the condition that it is constant among all knot diagrams for the knot. Since we can transform all knot diagrams for a given knot into any other knot diagram for the same knot, an equivalent definition is that a knot invariant does not change when we perform any Reidemeister move on a diagram of the knot.

Question 2.1. (3 pts) Is the number of crossings in the knot diagram of a given knot an invariant? That is, does the number of crossings of a knot diagram stay the same under Reidemeister moves? Prove that it does or give a counterexample.

An important feature of knot invariants is that we can use them to show that two knots are *not* equivalent. If two knots have different values of an invariant, we definitely can't transform one into the other.

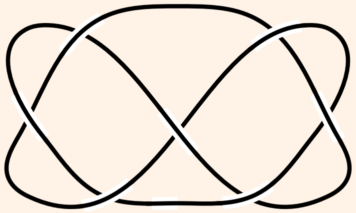
Definition 2.2. A **strand** in a knot diagram is a continuous line in the diagram that goes from one undercrossing (the string is below another part at a crossing) to another undercrossing, and any crossings in between are overcrossings (where the string passes above another part). In other words, a strand is a line in the diagram that can be drawn without lifting your pen, extending as far as it can in both directions.

Definition 2.3. A knot diagram is **tricolorable** if each of the strands in the diagram can be colored one of three different colors so that at least two colors are used in total and at each crossing, either three different colors come together or all the same colors come together.
When demonstrating tricolorability, you are welcome to use colors if you have them, or mark the color of each strand with numbers as demonstrated below.



Question 2.2. (2 pts) Show that if we did not require at least two colors to be used, every knot would be tricolorable.

Question 2.3. (3 pts) Show that the knot diagram below is tricolorable by coloring it.



Theorem 1.2 from Chapter 1 tells us that tricolorability is a knot invariant if it is maintained under each of the three Reidemeister moves.

Question 2.4. (4 pts) Prove that tricolorability is a knot invariant by showing it is invariant under Redemeister moves.

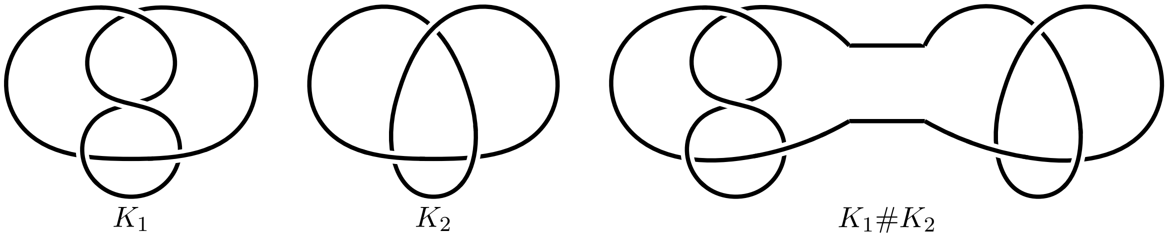
(a) Prove that tricolorability is invariant under type 1 Reidemeister moves.

(b) Prove that tricolorability is invariant under type 2 Reidemeister moves.

(c) Prove that tricolorability is invariant under type 3 Reidemeister moves.

Question 2.5. (3 pts) Using tricolorability, prove that the trefoil knot is not equivalent to the unknot.

Definition 2.4. Given two knot diagrams J and K, the **composition** $J\#K$ of two knot diagrams is obtained by removing a small arc from each diagram and connecting the four endpoints with two new arcs, as shown in the diagram below.

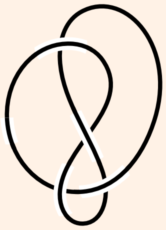


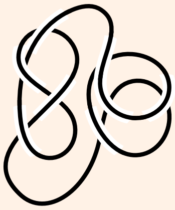
Question 2.6. (2 pts) Find a knot I such that for all K, $K\#I = K$. The knot I is called the identity element for the operation $\#$.

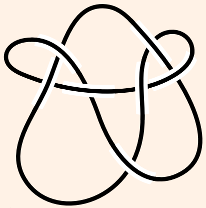
Question 2.7. (3 pts) Prove that the composition of two tricolorable knots is also tricolorable.

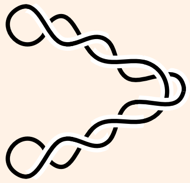
Tricolorability doesn't necessarily tell us whether two knot diagrams are equivalent, but in some cases it tells us that two diagrams are *definitely not equivalent*– if one is tricolorable and the other isn't.

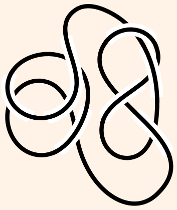
Question 2.8. (4 pts) Of the following five knot diagrams, exactly three are equivalent to each other and two are not. Determine which two are not equivalent to the rest.


(1)


(2)

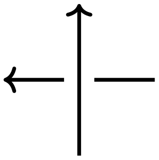

(3)

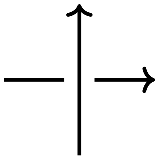

(4)


(5)

Now, let's look at another property of knot diagrams. If we pick a point on a knot diagram to start and then continuously trace the knot diagram in either direction (creating an oriented closed curve), each crossing will have a "direction" to it.

Definition 2.5. The **sign** of a crossing is 1 if the lower strand points right compared to the direction of the upper strand, and -1 if it points left.


Crossing with sign -1

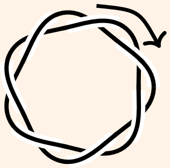

Crossing with sign 1

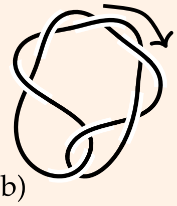
Definition 2.6. The **writhe** of a knot diagram is the sum of the signs of all crossings in the diagram.

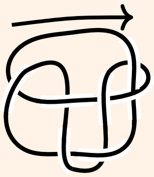
Let's explore writhe to see how it is useful.

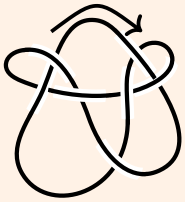
Question 2.9. (3 pts) Does the writhe of a diagram depend on which direction we trace the knot in? If so, how does it change? State and prove a claim.

Question 2.10. (4 pts) Compute the writhe of the following knot diagrams. If you determined that writhe depends on orientation, follow the knot in the direction as marked.


(a)


(b)


(c)


(d)

Let's conclude the section by determining whether writhe (in a clockwise direction, if it matters) is a knot invariant.

Question 2.11. (4 pts) For **each** of the three Reidemeister moves, prove or disprove the invariance of writhe. Then, conclude whether writhe is a knot invariant.

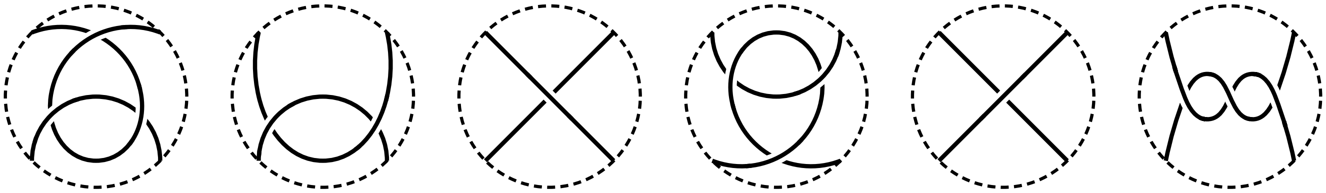
It turns out that writhe is very useful for the Kauffman polynomial, which gives rise to interesting invariants that we won't explore in-depth in this power round. Furthermore, we can adjust the definition of writhe to find an invariant for a generalization of knots called virtual knots.

3 Rational Knots (60 pts)

Let's now turn our focus to a specific type of knot called a *rational knot*. Rational knots are composed of smaller building blocks called *tangles*, which are diagrams of specific parts of knots.

Definition 3.1. A **tangle diagram** is a diagram contained in a disk, containing two strings with fixed endpoints on the boundary of the disk and possibly some crossings in the middle. For convenience, we place the fixed endpoints in the NW, NE, SE, and SW compass directions.

On the following page are some examples of tangle diagrams.



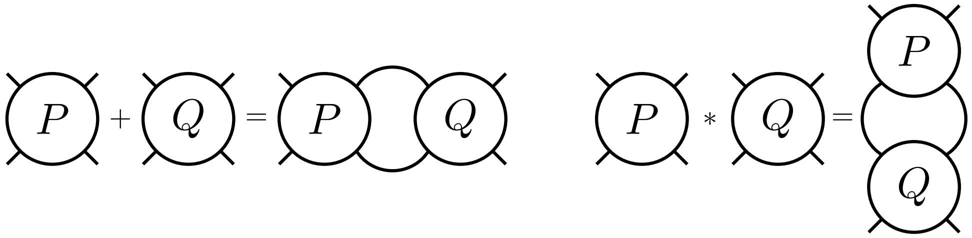
Question 3.1. (2 pts) Draw a tangle diagram with five crossings.

Definition 3.2. Two tangle diagrams P and Q are **equivalent** if one can be moved to look identical to the other using the Reidemeister moves provided in Chapter 1, while keeping the endpoints fixed. If P and Q are equivalent, we write $P \sim Q$.

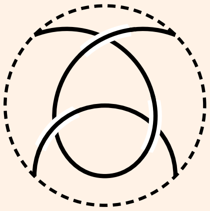
In order to analyze tangles, we first introduce some notation.

Definition 3.3. Given two tangles P and Q, we define:

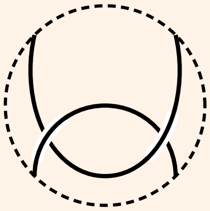
- The **sum**, $P + Q$, is the result of horizontally connecting P and Q by connecting the top right endpoint of P with the top left endpoint of Q, and similarly for the bottom points.
- The **product**, $P * Q$, is the result of vertically connecting P and Q and connecting adjacent endpoints.
- The **negation**, $-P$, is the result of switching all crossings in P (from ⤫ to ⤬ and vice versa).
- The **inverse**, P^{-1} or $\frac{1}{P}$, is the result of rotating P by 90 degrees clockwise and switching all crossings (taking the negation).
- The **difference** $P - Q \sim P + (-Q)$.



Question 3.2. (4 pts) Let T_1 and T_2 be the following tangles.



T_1

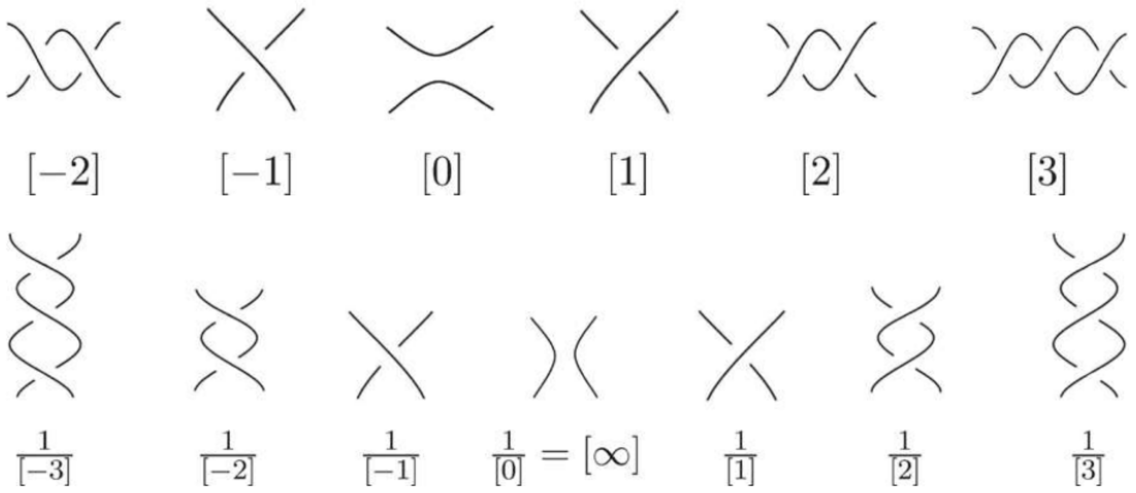


T_2

Draw:

(a) $T_1 + T_2$
(b) $T_1 * T_2$
(c) $-T_1$
(d) $\frac{1}{T_2}$

Definition 3.4. For nonnegative integers N, the **integer tangle** $[N]$ is the tangle with N horizontal crossings, with the first crossing being of type $\times$ and subsequent crossings alternating type as shown below. We define $[-N] = -[N]$. A **reciprocal tangle** is the inverse $\frac{1}{[N]}$ of an integer tangle. We define $\frac{1}{[0]} = [\infty]$ and $\frac{1}{[\infty]} = [0]$.



Integer and reciprocal tangles are the building blocks for our exploration of rational tangles!

Definition 3.5. A **rational tangle** is any tangle you can make by starting with an integer or reciprocal tangle and then repeating a finite number of steps where you either add an integer tangle or multiply by a reciprocal tangle on the left or right.

For example, $[1] + \frac{1}{[2]}$ is a rational tangle created by adding the integer tangle $[1]$ to $\frac{1}{[2]}$ on the left.

Theorem 3.6. For every rational tangle P, $(P^{-1})^{-1} = P$. In other words, taking the inverse twice gives the original tangle.

Note that in order to keep some nice properties of rational tangles, adding a reciprocal tangle to a rational tangle or multiplying a rational tangle by an integer tangle doesn't always result in a rational tangle.

Question 3.3. (5 pts) Prove the following statements about rational tangles.

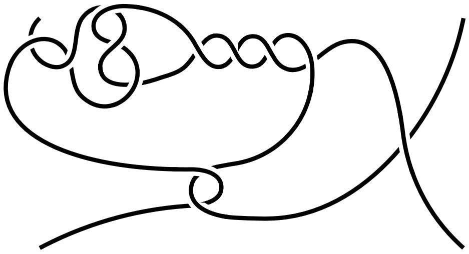
(a) For all integers m and n, $[m] + [n] \sim [m+n]$.

(b) For all integers m and n, $\frac{1}{[m]} * \frac{1}{[n]} \sim \frac{1}{[m]+[n]}$.

Question 3.4. (3 pts) Draw the following rational tangle:

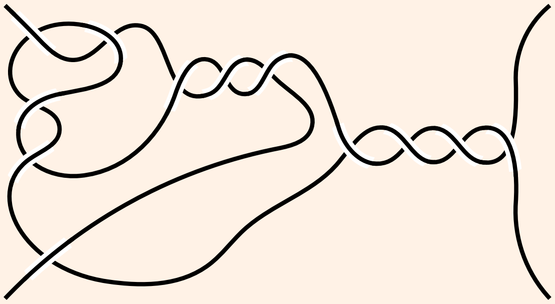
$$\left(\left([-1] * \frac{1}{[4]}\right) + [1]\right) * \frac{1}{[2]}.$$

Sometimes, it is not immediately obvious how to construct a rational tangle from integer and reciprocal tangles, as shown in the example below.



is $\left(\left([2] + \frac{1}{[3]} + [-4]\right) * \frac{1}{[-2]}\right) + [-1]$.

Question 3.5. (4 pts) Find a construction for the rational tangle shown below from integer and reciprocal tangles.



Question 3.6. (7 pts) For a rational tangle P, we define P^h to be the horizontal flip of P, obtained by rotating the tangle diagram around the horizontal axis 180°. Similarly, we define P^v to be the vertical flip of P, where we rotate along the vertical axis. Prove that $P \sim P^h$ and $P \sim P^v$. (Hint: use induction.)

Let's use our new notation to make some conclusions about when two tangles are equivalent.

Theorem 3.7. (Product-to-Inverse Equivalence): Given a rational tangle P and integer a,

$$P * \frac{1}{[a]} \sim \frac{1}{[a] + \frac{1}{P}}, \qquad \frac{1}{[a]} * P \sim \frac{1}{\frac{1}{P} + [a]}.$$

While this might seem at first to be an unnecessary observation, it turns out that being able to represent rational knots in a form that looks like a continued fraction can be very useful. This theorem allows us to change an arbitrary rational knot construction into a form that looks more like a continued fraction.

Question 3.7. (5 pts) Prove the Product-to-Inverse Equivalence by demonstrating an appropriate sequence of Reidemeister moves and diagrams to transform one side into the other.

In the last couple of chapters, we've spent a lot of time trying to figure out when two knots are equivalent. Using our new knowledge from this section, we can make a significant conclusion about when two rational tangles are equivalent.

First, our constructions of tangle diagrams like in Question 3.5 didn't necessarily look anything like a continued fraction. Let's find a standardized way of representing these tangles.

Definition 3.8. The **continued fraction form** of a rational tangle is a construction of a rational tangle using a continued fraction of integer tangles $[a_1], \ldots, [a_n]$ with $a_1 \neq 0$, as shown:

$$[a_n] + \cfrac{1}{[a_{n-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}}.$$

The **associated fraction** of the tangle is

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

Before we can use this form, we need to make sure that the form is well-defined– that is, each rational tangle has a (unique) continued fraction form. To see this, we can use the following theorem, which will also be very helpful in solving later problems in this chapter. Think about what it is saying for a second; it is a remarkable result that we unfortunately don't have time to prove during this round.

Theorem 3.9. For any rational tangle P and integer m,

$$P + [m] \sim [m] + P,$$

$$P * \frac{1}{[m]} \sim \frac{1}{[m]} * P.$$

Question 3.8. (6 pts) Given a construction of a rational tangle (a sequence of integer and rational tangles composed with the $+$ and $*$ operations, along with () to group terms), describe an algorithm to "translate" this construction into continued fraction form. (Hint: Use Product-to-Inverse Equivalence.)

By devising an appropriate algorithm, we may conclude that each tangle has a unique continued fraction form. Now that we know this is a valid representation of rational tangles, let's take our first look at our main theorem of this chapter!

Theorem 3.10. (Conway's Theorem): Two rational tangles are equivalent if and only if their associated fractions are equal.

This is a very powerful theorem, and we'll finish this chapter by working towards a proof of part of this statement. Let's start by thinking a bit more about continued fractions in general.

Question 3.9. (4 pts) Prove that for any fraction $\frac{p}{q} > 1$, there is a unique finite sequence of positive integers $a_1, \ldots, a_n$ such that $a_1 > 1$ and

$$\frac{p}{q} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

We also have the following formula:

$$a - \frac{1}{b} = (a-1) + \cfrac{1}{1 + \cfrac{1}{b-1}}$$

Amazingly, this formula is not just true for continued fractions, but also rational tangles!

Question 3.10. (12 pts) Prove that for any rational tangles P and Q,

$$P - \frac{1}{Q} \sim P - [1] + \cfrac{1}{[1] + \cfrac{1}{Q - [1]}}$$

by demonstrating an appropriate sequence of diagrams and/or Reidemeister moves to transform one side into the other.

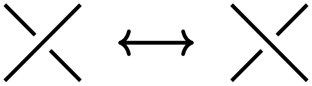
Now that we know this, we can prove the reverse direction of Conway's Theorem!

Question 3.11. (8 pts) Prove that if two rational tangles have equal associated fractions, then the tangles are equivalent. Remember that the fractions being equal doesn't mean that the continued fraction forms of the tangles are the same (since we don't necessarily have that each of the integer tangles is nonnegative)!

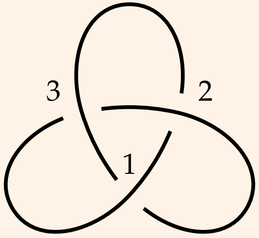
4 Unknotting Operations (79 pts)

As shown previously, many knot diagrams are equivalent to the unknot. More generally, we may take any knot, and by a series of operations, turn it into the unknot. One basic operation is the *crossing change*.

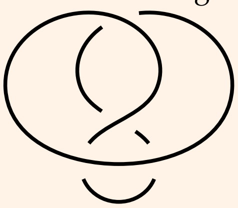
Definition 4.1. A **crossing change** is the operation of reversing which strand passes over the other at a single crossing. An example of a **crossing change** is shown below.

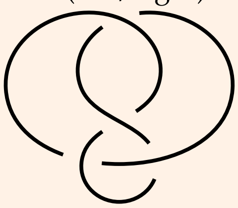


Question 4.1. (2 pts) Name a set of crossings in the knot diagram below that, when each changed, would result in the knot diagram being equivalent to the unknot.



Question 4.2. (2 pts) Which of the knot diagrams below (left/right) are equivalent to the unknot?





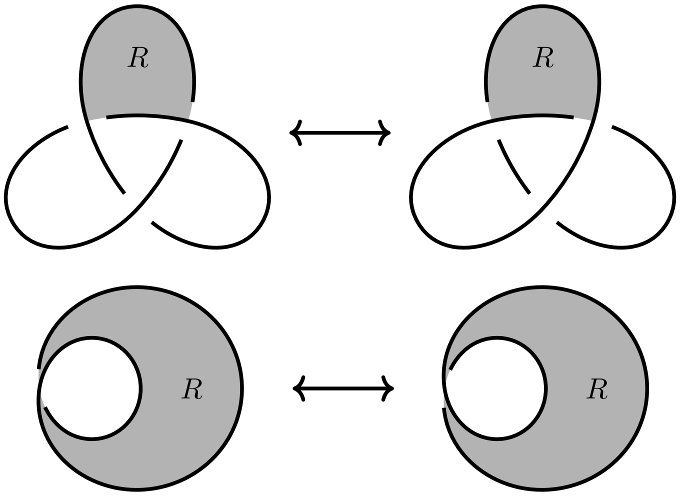
Question 4.3. (5 pts) Prove that in any knot diagram, there exists some set of crossing changes that can be made to make the diagram equivalent to the unknot.

Question 4.4. (3 pts) Prove that in any knot diagram, the minimum number of crossing changes needed to make the diagram equivalent to the unknot is at most half of the total number of crossings in the diagram.

Since any knot can be converted into the unknot through a series of crossing changes, the crossing change is called an *unknotting operation*. Another operation on knots is the *region crossing change*.

Definition 4.2. A **region crossing change** (RCC) is an operation that performs a crossing change on each crossing on the boundary of a region in a knot diagram. Note that the region outside of a knot is considered a valid region for RCC moves.

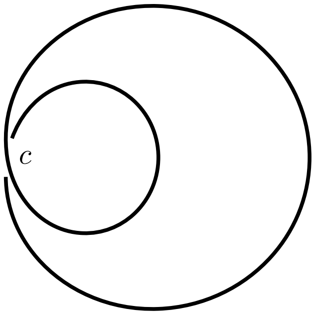
Examples of region crossing changes are shown below.



Question 4.5. (3 pts) Suppose that R and S are regions in a knot diagram. Prove that performing an RCC move on R followed by an RCC move on S produces the same modified knot diagram as performing an RCC move on S followed by an RCC move on R.

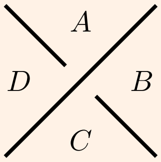
Notably, how RCCs affect the crossings of a diagram is related to how regions are positioned relative to crossings. In particular, crossings can be categorized by the regions around them.

Definition 4.3. A **reducible crossing** is a crossing where a region is adjacent to that crossing twice. For instance, crossing c is a reducible crossing in the knot diagram below. For the purposes of determining whether a crossing is reducible, the exterior of the knot is also considered a region.

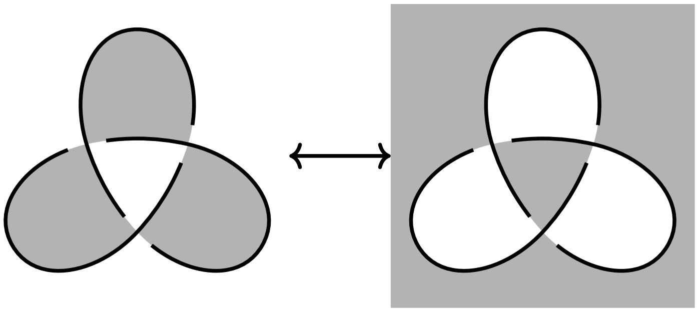


Definition 4.4. A **reduced diagram** is a knot diagram where none of the crossings are reducible.

Question 4.6. (2 pts) Suppose the crossing shown in the figure below is from an arbitrary knot diagram. Which regions (other than A) could be the same region as A?



Definition 4.5. A **checkerboard coloring** of a knot diagram is the shading of certain regions in the diagram such that unshaded regions are only adjacent to shaded regions, and shaded regions are only adjacent to unshaded regions. Two possible checkerboard colorings of a knot diagram are shown below. Note that the region outside the knot is also considered a region for the purposes of checkerboard coloring.



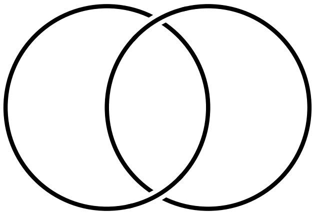
Question 4.7. (4 pts) Prove that checkerboard colorability is invariant under Reidemeister moves. In other words, if a Reidemeister move is performed somewhere on a checkerboard colorable knot diagram, the knot diagram can still be checkerboard colored. For this problem, you may not use the fact that all knot diagrams are checkerboard colorable, even if you are able to prove it independently.

Question 4.8. (5 pts) Prove that any knot diagram can be checkerboard colored.

Question 4.9. (3 pts) Consider a reduced knot diagram with a checkerboard coloring. Prove that the process of performing RCC moves on all shaded regions in the checkerboard coloring will result in the original reduced knot diagram.

For the rest of the section, we'll focus on proving that RCC is also an unknotting operation. In particular, given a knot diagram and any crossing c in the diagram, we can perform a series of RCCs so that c has its crossing changed, but no other crossings are changed. Since the crossing change is an unknotting operation, the RCC is also an unknotting operation.

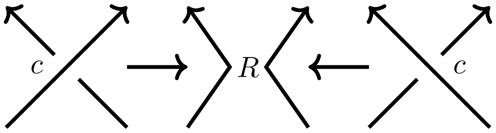
Definition 4.6. A **link** is a collection of closed loops. Each closed loop in the link is a **component** of the link. For example, the link diagram below has two components.



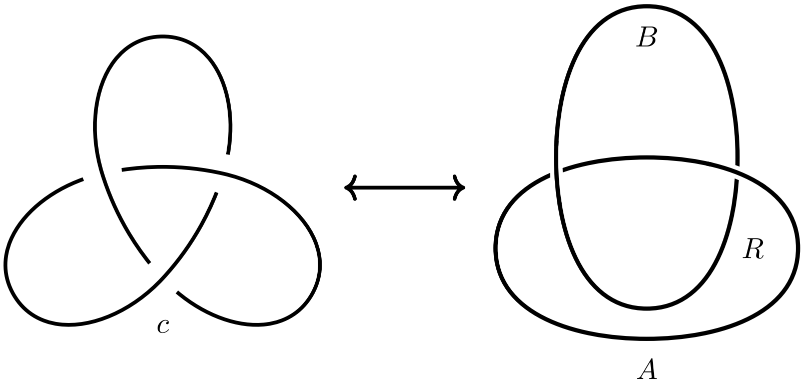
Recall from Chapter 2 that we can trace along a knot in a particular direction. We can use arrows to denote the direction we are moving along the knot in. This notion of direction is also useful for unknotting, as we see below.

Definition 4.7. A **splice** is the act of removing a crossing from an oriented knot diagram with respect to the orientation of the diagram. Examples of splices are shown below. Note that the orientation of the diagram should be preserved.

The diagram below shows a splice on crossing c. Region R is the region where crossing c was located at before the splice. Note that the arrows denoting orientation are in their original positions after the splice.



A splice on crossing c in the trefoil is shown below. Region R is the region after the splice where crossing c used to be. Note that after the splice, the trefoil is split into a link diagram with components A and B.

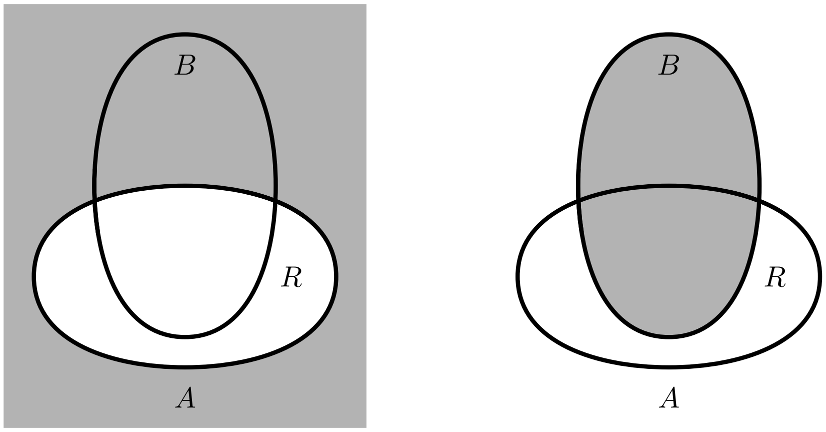


Question 4.10. (4 pts) Prove that a splice on a knot diagram will always convert it to a link diagram with two links.

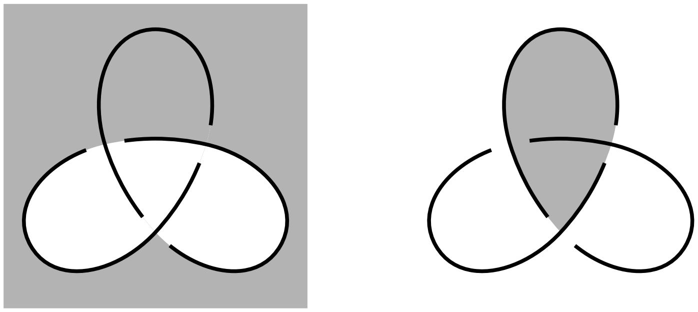
It is not obvious how to determine the regions on which we should perform RCCs. To start, let's consider the following three-step operation:

1. Splice a crossing c in the knot diagram. Let R denote the connected region from this splice.
2. On the resulting diagram, checkerboard color one of the two components while ignoring the other. Choose the checkerboard coloring that does not shade in the region R.

   The diagrams below show two ways to checkerboard color the spliced diagram of the trefoil. The diagram on the left shows when component A is colored. The diagram on the right shows when component B is colored. Note that region R is unshaded in both cases.



3. Undo the splice, remembering which regions were shaded in the checkerboard coloring. Then, perform an RCC move in all of the shaded regions in the unspliced knot diagram. For instance, the below diagrams depict the unspliced versions of the links from the examples above. Note that region crossing changes will be performed on the shaded regions.



Question 4.11. (8 pts) Prove that performing the three-step operation above on crossing c in a knot diagram is equivalent to a crossing change on c if all crossings other than c are not reducible.

As shown above, the three-step operation can be used to successfully unknot any reduced knot diagram. Now, we consider knot diagrams that still have reducible knots.

Question 4.12. (6 pts) Suppose a knot diagram D has exactly one reducible crossing, d. Prove that for any other crossing, c, a series of RCC moves exists that results in a crossing change on c and no other crossings.

For knot diagrams with more than 1 reducible crossing, we can prove that there exists a series of RCC moves equivalent to any crossing change through induction on the number of reducible crossings. We have already proven the base case above, which is the case of diagrams with zero reducible crossings.

Suppose we are given that for some nonnegative integer k, for knot diagrams with up to k reducible crossings, for any crossing c, there exists a series of RCC moves that will cause a crossing change at c without changing any other crossings. We now want to prove the case for up to $k+1$ reducible crossings.

Consider an arbitrary knot diagram with $k+1$ reducible crossings. We want to perform a crossing change on c (which may or may not be reducible) through a series of RCC moves without changing any other crossings.

Question 4.13. (4 pts) Prove that if a knot diagram is spliced at a reducible crossing, the two resulting links do not have any intersections in the diagram.

Question 4.14. (12 pts) Prove that for all integers $k \geq 1$, on any knot diagram with $k+1$ reducible crossings, there exists a reducible crossing $p \neq c$ such that splicing at p will result in a component with c and k reducible crossings, and another reduced component that does not include c or any reducible crossings. (Hint: First prove that there is a crossing p where the diagram can be spliced into one component with k reducible crossings and one component with zero reducible crossings, without considering where c is. Then, if c is not on the component with k reducible crossings, devise a way to find a different reducible crossing q where c is on the component with k reducible crossings.)

Question 4.15. (16 pts) Prove that for any crossing c on any knot diagram with $k+1$ reducible crossings (where $k \geq 1$), there exists a series of RCC moves that changes the crossing on c and does not change any other crossings, assuming that this statement holds for all knot diagrams with k reducible crossings.

Then by induction, on any knot diagram with an arbitrary number of reducible crossings, there exists a set of RCC moves to make a crossing change on any crossing c and no other crossings. So, we can unknot any knot diagram with RCC moves by first finding the set of crossing changes needed, and then performing RCC moves to get those crossing changes. In other words, we have found that RCCs are simple but surprisingly powerful operations for transforming knots.

Congratulations on finishing the BMT 2025 Power Round! We hope that you enjoyed exploring knot theory and are excited to learn more about it in the future. Good luck on your next tests, and have fun!